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Electromagnetic radiation in non-thermal quantum plasmas

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Abstract. Green function theory is used to describe the radiation spectrum of non-thermal quantum plasmas. A general formula is derived from a statistical average of the Poynting vector and a formal analogy with the equilibrium results is established in the case of slowly varying disturbances.

1. Introduction

The statistical average of the Poynting vector has been widely used to calculate the flow of energy carried by electromagnetic fields in classical plasmas (see, e.g., Ichimaru *et al* [1], Fidone *et al* [2] among many authors). Our aim was to extend this method to non-equilibrium quantum plasmas, and the need for rigour led us to the choice of a formalism that permits us to treat the charged particles and the radiation field on the same footing.

The theoretical framework adopted here is the non-equilibrium Green function technique, initially due to Kadanoff and Baym [3], applied to the study of the statistical properties of radiation in a plasma [4-6]. A brief account of this theory is given in § 2. In § 3 the averaged Poynting vector is expressed in terms of the real-time correlation functions $D^{\gtrless}_{\mu\nu}$. Then, introducing a set of local variables, we show that the radiation spectrum can be evaluated through the use of generalised distribution functions $N_{\mu\nu}$ and spectral functions $\Delta_{\mu\nu}$ just as in equilibrium. In § 4 the plasma is considered to be locally isotropic and the external disturbances to be slowly varying in time and space. In such a case our general result reduces to an expression where the transverse dielectric function of the medium appears explicitly. We conclude this work by showing how the radiation spectrum for a plasma in thermodynamic equilibrium can be easily derived from this formalism.

2. Statistical description of radiation field

We consider a system of charged particles interacting with the electromagnetic field. The statistical state of this system is described by the Hamiltonian H_0 and the (equilibrium) density matrix ρ_0. At the time t_0, externally controlled disturbances (c-number currents $j^{\text{ext}}_{\mu}(t)$ [7]) drive the system away from its initial equilibrium state. The full Hamiltonian is now

$$H(t) = H_0 + H_{\text{ext}}(t) \tag{1}$$

and the ensemble average, for any observable X, may be written as [3-6]

$$\langle X(t)\rangle = \mathrm{Tr}[\rho_0 X(t)] \qquad t > t_0 \tag{2}$$

where the Heisenberg picture operator $X(t)$ is related to the interaction operator $X_\mathrm{I}(t)$ through

$$X(t) = S^+(t, t_0)X_\mathrm{I}(t)S(t, t_0) \tag{3}$$

with

$$S(t, t_0) = T\left(\exp(\mathrm{i}\hbar)^{-1}\int_{t_0}^{t}\mathrm{d}t' j_\nu^{\mathrm{ext}}(t')A_\mathrm{I}^\nu(t')\right). \tag{4}$$

T denotes the usual Wick chronological operator and A^ν the 4-potential operator (here and below greek indices take the values 0, 1, 2, 3 while latin ones take the values 1, 2, 3).

Now we can introduce the various Green functions used throughout this paper. The time-ordered Green function $D_{\mu\nu}(x^\lambda, x'^\lambda)$ is defined by

$$D_{\mu\nu}(x^\lambda, x'^\lambda) = (\mathrm{i}\hbar\mu_0)^{-1}\langle T[\hat{A}_\mu(x^\lambda)\hat{A}_\nu(x'^\lambda)]\rangle \tag{5}$$

where $x^\lambda = (ct, \boldsymbol{x})$ and $\hat{A}_\mu = A_\mu - \langle A_\mu\rangle$. This quantity is nothing more than a generalisation of the propagator used in vacuum quantum electrodynamics [8]. We note a difference in sign between the 'statistical' propagator and the 'vacuum' one: in (5) the sign is chosen in order to correspond to the definition of the Green functions of other bosons (e.g. phonons). This propagator may also be expressed in terms of the correlation function $D_{\mu\nu}^{\gtrless}(x^\lambda, x'^\lambda)$:

$$D_{\mu\nu}(x^\lambda, x'^\lambda) = \Theta(x_0 - x_0')D_{\mu\nu}^{>}(x^\lambda, x'^\lambda) + \Theta(x_0' - x_0)D_{\mu\nu}^{<}(x^\lambda, x'^\lambda) \tag{6}$$

where Θ is the Heaviside function.

Another related quantity is the response function (i.e. the retarded propagator) $D_{\mu\nu}^{\mathrm{R}}(x^\lambda, x'^\lambda)$:

$$D_{\mu\nu}^{\mathrm{R}}(x^\lambda, x'^\lambda) = \Theta(x_0 - x_0')[D_{\mu\nu}^{>}(x^\lambda, x'^\lambda) - D_{\mu\nu}^{<}(x^\lambda, x'^\lambda)] \tag{7}$$

which satisfies the Dyson equation, just as in the equilibrium theory,

$$D_{\mu\nu}^{\mathrm{R}}(x^\lambda, x'^\lambda) = D_{\mu\nu}^{0\mathrm{R}}(x^\lambda - x'^\lambda) + \int \mathrm{d}y^\lambda\, \mathrm{d}y'^\lambda\, D_{\mu\gamma}^{0\mathrm{R}}(x^\lambda - y^\lambda)Q_\mathrm{R}^{\gamma\sigma}(y^\lambda, y'^\lambda)D_{\sigma\nu}^{\mathrm{R}}(y'^\lambda, x'^\lambda). \tag{8}$$

The polarisation tensor $Q_{\mu\nu}^{\mathrm{R}}$ describes the influence of the medium on the propagation of electromagnetic waves, including all the effects responsible for collective modes and screening in the plasma. In fact, as we shall see later, this function is closely related to the dielectric tensor of the plasma; the rates of the processes contributing to wave damping (principally Laudau damping and, at high densities and temperatures, pair production) may thus be calculated from the imaginary part of $Q_{\mu\nu}^{\mathrm{R}}$.

Since the knowledge of $D_{\mu\nu}^{>}$ and $D_{\mu\nu}^{<}$ gives us rather complete statistical information on the fields, the Green function technique appears to be a useful basis for the study of radiation spectra in non-equilibrium plasmas.

3. The Poynting vector and radiation spectrum

The symmetrised Poynting vector is given by

$$\boldsymbol{S}(x^\lambda) = (1/2\mu_0)[\boldsymbol{E}(x^\lambda) \wedge \boldsymbol{B}(x^\lambda) - \boldsymbol{B}(x^\lambda) \wedge \boldsymbol{E}(x^\lambda)]. \tag{9}$$

In terms of the electromagnetic field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, equation (9) becomes

$$S^i(x^\lambda) = (c/2\mu_0)[F_{0\mu}(x^\lambda)F^{\mu i}(x^\lambda) + F^{i\mu}(x^\lambda)F_{\mu 0}(x^\lambda)]. \tag{10}$$

Instead of $\boldsymbol{E}$ and $\boldsymbol{B}$, we now use the fluctuating quantities $\hat{\boldsymbol{E}} = \boldsymbol{E} - \langle \boldsymbol{E}\rangle$ and $\hat{\boldsymbol{B}} = \boldsymbol{B} - \langle \boldsymbol{B}\rangle$. The statistical average of the flow of energy carried by these fields may thus be written

$$\langle \hat{S}^i(x^\lambda)\rangle = (c/2\mu_0)\langle \hat{F}_{0\mu}(x^\lambda)\hat{F}^{\mu i}(x^\lambda) + \hat{F}^{i\mu}(x^\lambda)\hat{F}_{\mu 0}(x^\lambda)\rangle \tag{11}$$

where $\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu$. Hence

$$\begin{aligned}\langle \hat{S}^i(x^\lambda)\rangle = (c/2\mu_0) \lim_{x'^\lambda \to x^\lambda} \langle & \partial_\mu \partial'^i[\hat{A}_0(x^\lambda)\hat{A}^\mu(x'^\lambda) + \hat{A}^\mu(x'^\lambda)\hat{A}_0(x^\lambda)] \\ & - \partial_\mu \partial'^\mu[\hat{A}_0(x^\lambda)\hat{A}^i(x'^\lambda) + \hat{A}^i(x'^\lambda)\hat{A}_0(x^\lambda)] \\ & + \partial_0 \partial'^\mu[\hat{A}_\mu(x^\lambda)\hat{A}^i(x'^\lambda) + \hat{A}^i(x'^\lambda)\hat{A}_\mu(x^\lambda)] \\ & - \partial_0 \partial'^i[\hat{A}_\mu(x^\lambda)\hat{A}^\mu(x'^\lambda) + \hat{A}^\mu(x'^\lambda)\hat{A}_\mu(x^\lambda)]\rangle. \end{aligned} \tag{12}$$

The limit must be taken after the derivatives have been calculated.

In terms of the correlation functions $D^{\gtrless}_{\mu\nu}(x^\lambda, x'^\lambda)$, we thus obtain

$$\begin{aligned}\langle \hat{S}^i(x^\lambda)\rangle = (\mathrm{i}\hbar c/2) \lim_{x'^\lambda \to x^\lambda} \langle & \partial_\mu \partial'^i[D_0^{>\mu}(x^\lambda, x'^\lambda) + D_0^{<\mu}(x^\lambda, x'^\lambda)] \\ & - \partial_\mu \partial'^\mu[D_0^{>i}(x^\lambda, x'^\lambda) + D_0^{<i}(x^\lambda, x'^\lambda)] + \partial_0 \partial'^\mu[D_\mu^{>i}(x^\lambda, x'^\lambda) + D_\mu^{<i}(x^\lambda, x'^\lambda)] \\ & - \partial_0 \partial'^i[D_\mu^{>\mu}(x^\lambda, x'^\lambda) + D_\mu^{<\mu}(x^\lambda, x'^\lambda)]\rangle. \end{aligned} \tag{13}$$

In order to write this equation in a local form (see, for example, the clear and recent review by Henneberger *et al* [9]) we use a new set of coordinates (r^λ, R^λ) with $r^\lambda = x^\lambda - x'^\lambda$ and $R^\lambda = (x^\lambda + x'^\lambda)/2$. Then, the correlation functions $D^{\gtrless}_{\mu\nu}(r^\lambda, R^\lambda)$ are expressed in terms of their Fourier transforms (with respect to the difference variable r^λ) according to

$$D^{\gtrless}_{\mu\nu}(r^\lambda, R^\lambda) = (2\pi)^{-4} \int \mathrm{d}k^\lambda \, cD^{\gtrless}_{\mu\nu}(k^\lambda, R^\lambda) \exp(-\mathrm{i}k^\lambda r_\lambda) \tag{14a}$$

$$cD^{\gtrless}_{\mu\nu}(k^\lambda, R^\lambda) = \int \mathrm{d}r^\lambda \, D^{\gtrless}_{\mu\nu}(r^\lambda, R^\lambda) \exp(\mathrm{i}k^\lambda r_\lambda) \qquad k^\lambda = (\omega/c, \hbar^{-1}\boldsymbol{p}). \tag{14b}$$

The range of values allowed for k_0 is thus $]-\infty; +\infty[$. Nevertheless, the frequency is customarily considered as a positive quantity, so, in order to deal with only positive k_0, we employ the real quantity $\mathrm{i}D^{\gtrless}_{\mu\nu}(k^\lambda, R^\lambda)$ to restate (14*a*) as

$$\mathrm{i}D^{\gtrless}_{\mu\nu}(r^\lambda, R^\lambda) = [2/(2\pi)^4] \operatorname{Re} \int \mathrm{d}k^\lambda \, \mathrm{i}c\, D^{\gtrless}_{\mu\nu}(k^\lambda, R^\lambda) \exp(-\mathrm{i}k^\lambda r_\lambda) \tag{14c}$$

where (except with specific indication) the range of integration over $\mathrm{d}k_0$ has to be understood henceforth as $[0; +\infty[$.

Performing these changes in (13) we find

$$\begin{aligned}\langle \hat{S}^i(x^\lambda)\rangle = [\hbar c^2/(2\pi)^4] \operatorname{Re} \int \mathrm{d}k^\lambda \, \mathrm{i}\{ & k_j k^i[D_0^{>j}(k^\lambda, x^\lambda) + D_0^{<j}(k^\lambda, x^\lambda)] \\ & + k_0 k^j[D_j^{>i}(k^\lambda, x^\lambda) + D_j^{<i}(k^\lambda, x^\lambda)] + k^2[D_0^{>i}(k^\lambda, x^\lambda) + D_0^{<i}(k^\lambda, x^\lambda)] \\ & - k_0 k^i[D_j^{>j}(k^\lambda, x^\lambda) + D_j^{<j}(k^\lambda, x^\lambda)]\}. \end{aligned} \tag{15}$$

The local coordinates allow us to introduce two useful (real) functions: the spectral function $\Delta_{\mu\nu}(k^\lambda, R^\lambda)$ and the generalised 'distribution function' $N_{\mu\nu}(k^\lambda, R^\lambda)$ [4-6].

Let us recall first some elements of the equilibrium theory. In an equilibrium isotropic system, the Green functions depend on the difference variables only (i.e. $D^{\mathrm{eq}}_{\mu\nu}(x^\lambda, x'^\lambda) = D_{\mu\nu}(r^\lambda)$ and satisfy the famous Kubo-Martin-Schwinger boundary condition. In Fourier space, we thus have

$$D^{>}_{\mu\nu}(k^\lambda) = \exp(\beta\hbar c k_0) D^{<}_{\mu\nu}(k^\lambda) \qquad \beta = 1/k_{\mathrm{B}}T \tag{16}$$

and the spectral function $\Delta^{\mathrm{eq}}_{\mu\nu}(k^\lambda)$ is defined by

$$\Delta^{\mathrm{eq}}_{\mu\nu}(k^\lambda) = \mathrm{i}[D^{>}_{\mu\nu}(k^\lambda) - D^{<}_{\mu\nu}(k^\lambda)]. \tag{17}$$

A spectral representation of $D^{\mathrm{R}}_{\mu\nu}(k^\lambda)$ is then possible (the so-called Lehmann-Källen spectral representation) through the Fourier transform of the equilibrium version of (7).

Using

$$\int_{-\infty}^{+\infty} \mathrm{d}r_0\, \Theta(r_0) \exp(\mathrm{i}k_0 r_0) = \mathrm{i}/(k_0 + \mathrm{i}\eta) \qquad \eta \to 0^+ \tag{18}$$

we obtain the spectral representation

$$\begin{aligned} D^{\mathrm{R}}_{\mu\nu}(k^\lambda) &= (2\pi)^{-1} \int_{-\infty}^{+\infty} \mathrm{d}k'_0\, \Delta^{\mathrm{eq}}_{\mu\nu}(k'_0, k^i)/(k_0 - k'_0 + \mathrm{i}\eta) \\ &= (2\pi)^{-1}\, \mathrm{PV} \int_{-\infty}^{+\infty} \mathrm{d}k'_0\, \Delta^{\mathrm{eq}}_{\mu\nu}(k'_0, k^i)/(k_0 - k'_0) - \mathrm{i}\Delta^{\mathrm{eq}}_{\mu\nu}(k^\lambda)/2. \end{aligned} \tag{19}$$

PV denotes the principal value of the integral. Let us recall that $\Delta^{\mathrm{eq}}_{\mu\nu}$ is the quantity which connects the imaginary-time propagator, computable from the standard perturbation expansion, to the physical (retarded) one [3, 4, 10]. Thus, from (19)

$$\Delta^{\mathrm{eq}}_{\mu\nu}(k^\lambda) = -2\, \mathrm{Im}\, D^{\mathrm{R}}_{\mu\nu}(k^\lambda). \tag{20}$$

We easily verify that the boundary condition on $D_{\mu\nu}(k^\lambda)$ can be restated as

$$D^{>}_{\mu\nu}(k^\lambda) = -\mathrm{i}[g_{\mu\alpha} + N_{\mu\alpha}(k_0)]\Delta^{\alpha}_{\mathrm{eq}\,\nu}(k^\lambda) \tag{21a}$$

$$D^{<}_{\mu\nu}(k^\lambda) = -\mathrm{i}N_{\mu\alpha}(k_0)\Delta^{\alpha}_{\mathrm{eq}\,\nu}(k^\lambda) \tag{21b}$$

where we write for the generalised (equilibrium) distribution function

$$N_{\mu\alpha}(k_0) = g_{\mu\alpha} N^{\mathrm{eq}}(k_0). \tag{22a}$$

$N^{\mathrm{eq}}(k_0)$ is the Bose-Einstein distribution

$$N^{\mathrm{eq}}(k_0) = [\exp(\beta\hbar c k_0) - 1]^{-1} \tag{22b}$$

and the metric is $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$.

In the non-equilibrium case, the correlation functions no longer satisfy (16) but we can define, in a similar manner, a spectral function $\Delta_{\mu\nu}(k^\lambda, R^\lambda)$:

$$\Delta_{\mu\nu}(k^\lambda, R^\lambda) = \mathrm{i}[D^{>}_{\mu\nu}(k^\lambda, R^\lambda) - D^{<}_{\mu\nu}(k^\lambda, R^\lambda)]. \tag{23}$$

Just as in equilibrium, the local form of (7) leads straightforwardly to a spectral representation analogous to (19). Hence

$$\Delta_{\mu\nu}(k^\lambda, R^\lambda) = -2\, \mathrm{Im}\, D^{\mathrm{R}}_{\mu\nu}(k^\lambda, R^\lambda). \tag{24}$$

Taking (23) into account we now introduce the so-called generalised distribution function, $N_{\mu\nu}(k^\lambda, R^\lambda)$, by writing the non-equilibrium counterpart of (21) as (see [6, p 40])†

$$D^{>}_{\mu\nu}(k^\lambda, R^\lambda) = -\mathrm{i}[g_{\mu\alpha} + N_{\mu\alpha}(k^\lambda, R^\lambda)]\Delta^{\alpha}{}_{\nu}(k^\lambda, R^\lambda) \tag{25a}$$

$$D^{<}_{\mu\nu}(k^\lambda, R^\lambda) = -\mathrm{i}N_{\mu\alpha}(k^\lambda, R^\lambda)\Delta^{\alpha}{}_{\nu}(k^\lambda, R^\lambda). \tag{25b}$$

Obviously, $N_{\mu\nu}(k^\lambda, R^\lambda)$ is now an unknown quantity.

Taking the sum of (25a) and (25b) we obtain

$$\mathrm{i}[D^{>}_{\mu\nu}(k^\lambda, R^\lambda) + D^{<}_{\mu\nu}(k^\lambda, R^\lambda)] = [g_{\mu\alpha} + 2N_{\mu\alpha}(k^\lambda, R^\lambda)]\Delta^{\alpha}{}_{\nu}(k^\lambda, R^\lambda) \tag{26}$$

and the averaged Poynting vector becomes

$$\langle \hat{S}^i(x^\lambda)\rangle = [\hbar c^2/(2\pi)^4] \int \mathrm{d}k^\lambda [k^i \Delta^{\mu j}(k^\lambda, x^\lambda) - k^j \Delta^{\mu i}(k^\lambda, x^\lambda)]$$
$$\times \{k_j[g_{0\mu} + 2N_{0\mu}(k^\lambda, x^\lambda)] - k_0[g_{j\mu} + 2N_{j\mu}(k^\lambda, x^\lambda)]\}. \tag{27}$$

We now look for the frequency spectrum of the electromagnetic power carried to the point x^λ, striking an infinitesimal area $\mathrm{d}\Sigma$ (with a unit normal $\boldsymbol{n}$) into a solid angle $\mathrm{d}\Omega$ in the direction of $\boldsymbol{k}$.

Writing $\langle \hat{\boldsymbol{S}}(x^\lambda)\rangle \cdot \boldsymbol{n}$ as $\langle \hat{S}^i(x^\lambda)\rangle n^i$ and expanding $\int \mathrm{d}k^\lambda$ as $\int_0^\infty \mathrm{d}k_0 \int \mathrm{d}\Omega \int_0^\infty \mathrm{d}k\, k^2$ we obtain

$$\mathrm{d}P(x^\lambda)/\mathrm{d}k_0\, \mathrm{d}\Omega\, \mathrm{d}\Sigma = [\hbar c^2/(2\pi)^4] \int_0^\infty \mathrm{d}k\, k^2 [k^i \Delta^{\mu j}(k^\lambda, x^\lambda) - k^j \Delta^{\mu i}(k^\lambda, x^\lambda)]$$
$$\times \{k_j[g_{0\mu} + 2N_{0\mu}(k^\lambda, x^\lambda)] - k_0[g_{j\mu} + 2N_{j\mu}(k^\lambda, x^\lambda)]\} n^i. \tag{28}$$

Equation (28) is our main result; it describes, in a fully general manner, the radiation spectrum of non-equilibrium plasmas.

4. Slowly varying disturbances

In this section we consider the externally driven sources to be slowly varying in time and space [3-6], i.e. all the quantities introduced in the previous sections are slowly varying functions of R^λ on a microscopic scale determined by the range in r^λ (which depends explicitly on the physical problem studied). As noted in [5] one should not confuse this approximation with the hydrodynamic local equilibrium limit, which requires a still lower variation.

In our case, a gradient expansion applied to the local form of (8) yields

$$D^{\mathrm{R}}_{\mu\nu}(k^\lambda, R^\lambda) = D^{0\mathrm{R}}_{\mu\nu}(k^\lambda) + D^{0\mathrm{R}}_{\mu\gamma}(k^\lambda) Q^{\gamma\sigma}_{\mathrm{R}}(k^\lambda, R^\lambda) D^{\mathrm{R}}_{\sigma\nu}(k^\lambda, R^\lambda). \tag{29}$$

Henceforth, using ordinary 3D notation $(i, j, k = x, y, z)$ the assumption of a locally isotropic medium leads to [6]

$$Q^{\mathrm{R}}_{ij}(k^\lambda, R^\lambda) = (\delta_{ij} - k_i k_j/k^2) Q^{\mathrm{R}}_{\mathrm{T}}(k^\lambda, R^\lambda) + (k_i k_j/k^2) Q^{\mathrm{R}}_{\mathrm{L}}(k^\lambda, R^\lambda) \tag{30a}$$

$$Q^{\mathrm{R}}_{i0}(k^\lambda, R^\lambda) = Q^{\mathrm{R}}_{0i}(k^\lambda, R^\lambda) = (k_0 k_i/k^2) Q^{\mathrm{R}}_{00}(k^\lambda, R^\lambda) \tag{30b}$$

$$Q^{\mathrm{R}}_{\mathrm{L}}(k^\lambda, R^\lambda) = -(k_0/k)^2 Q^{\mathrm{R}}_{00}(k^\lambda, R^\lambda) \tag{30c}$$

† The tensorial feature of the Green functions does not appear in their equation (5.3) because the authors used a complete set of real eigenvectors to diagonalise them.

where $Q_{\mathrm{T}}^{\mathrm{R}}(k^\lambda, R^\lambda)$ and $Q_{\mathrm{L}}^{\mathrm{R}}(k^\lambda, R^\lambda)$ are the transverse and longitudinal projections of $Q_{ij}^{\mathrm{R}}(k^\lambda, R^\lambda)$.

All the equations written in this section are gauge invariant. A particular gauge is established by calculating $D_{\mu\nu}^{0\mathrm{R}}(k^\lambda)$ in that gauge. In the Coulomb gauge we have the well known results

$$D_{00}^{0\mathrm{R}}(k^\lambda) = 1/[k^2 + \mathrm{i}\,\mathrm{sgn}(k_0)\eta] \qquad D_{0i}^{0\mathrm{R}}(k^\lambda) = D_{i0}^{0\mathrm{R}}(k^\lambda) = 0 \tag{31a}$$

$$D_{\mathrm{T}}^{0\mathrm{R}}(k^\lambda) = -1/[k^2 - (k_0 + \mathrm{i}\eta)^2] \qquad D_{ij}^{0\mathrm{R}}(k^\lambda) = D_{\mathrm{T}}^{0\mathrm{R}}(k^\lambda)(\delta_{ij} - k_ik_j/k^2). \tag{31b}$$

If we substitute these propagators into (29) we obtain

$$D_{00}^{\mathrm{R}}(k^\lambda, R^\lambda) = D_{00}^{0\mathrm{R}}(k^\lambda)/[1 - D_{00}^{0\mathrm{R}}(k^\lambda)Q_{00}^{\mathrm{R}}(k^\lambda, R^\lambda)] \tag{32a}$$

$$D_{0i}^{\mathrm{R}}(k^\lambda, R^\lambda) = D_{i0}^{\mathrm{R}}(k^\lambda, R^\lambda) = 0 \tag{32b}$$

$$D_{\mathrm{T}}^{\mathrm{R}}(k^\lambda, R^\lambda) = D_{\mathrm{T}}^{0\mathrm{R}}(k^\lambda)/[1 - D_{\mathrm{T}}^{0\mathrm{R}}(k^\lambda)Q_{\mathrm{T}}^{\mathrm{R}}(k^\lambda, R^\lambda)] \tag{32c}$$

$$D_{ij}^{\mathrm{R}}(k^\lambda, R^\lambda) = D_{\mathrm{T}}^{\mathrm{R}}(k^\lambda, R^\lambda)(\delta_{ij} - k_ik_j/k^2). \tag{32d}$$

Introducing the transverse and longitudinal dielectric functions

$$\varepsilon_{\mathrm{T,L}}(k^\lambda, R^\lambda) = 1 + Q_{\mathrm{T,L}}^{\mathrm{R}}(k^\lambda, R^\lambda)/k_0^2 \tag{33}$$

equations (32a) and (32c) become

$$D_{00}^{\mathrm{R}}(k^\lambda, R^\lambda) = 1/k^2\varepsilon_{\mathrm{L}}(k^\lambda, R^\lambda) \tag{34a}$$

$$D_{\mathrm{T}}^{\mathrm{R}}(k^\lambda, R^\lambda) = 1/[k_0^2\varepsilon_{\mathrm{T}}(k^\lambda, R^\lambda) - k^2] \tag{34b}$$

which are similar to the equilibrium ones [10]. In this simplified case (25a) and (25b) take the form (see [5, pp 552, 570])

$$D_{00}^{>}(k^\lambda, R^\lambda) = -\mathrm{i}[1 + N_{00}(k^\lambda, R^\lambda)]\Delta_{00}(k^\lambda, R^\lambda) \tag{35a}$$

$$D_{00}^{<}(k^\lambda, R^\lambda) = -\mathrm{i}\ N_{00}(k^\lambda, R^\lambda)\Delta_{00}(k^\lambda, R^\lambda) \tag{35b}$$

$$D_{\mathrm{T}}^{>}(k^\lambda, R^\lambda) = -\mathrm{i}[1 + N_{\mathrm{T}}(k^\lambda, R^\lambda)]\Delta_{\mathrm{T}}(k^\lambda, R^\lambda) \tag{35c}$$

$$D_{\mathrm{T}}^{<}(k^\lambda, R^\lambda) = -\mathrm{i}\ N_{\mathrm{T}}(k^\lambda, R^\lambda)\Delta_{\mathrm{T}}(k^\lambda, R^\lambda). \tag{35d}$$

N_{00} and N_{T} represent, respectively, the plasmon and transverse photon generalised distribution functions.

We now turn back to (28) and limit ourselves to the radiation perpendicular to the area $\mathrm{d}\Sigma$. We easily find that it reduces to

$$\mathrm{d}P(x^\lambda)/\mathrm{d}k_0\,\mathrm{d}\Omega\,\mathrm{d}\Sigma = (\hbar c^2k_0/8\pi^4)\int_0^\infty \mathrm{d}k\,k^3[1 + 2N_{\mathrm{T}}(k^\lambda, x^\lambda)]\Delta_{\mathrm{T}}(k^\lambda, x^\lambda). \tag{36}$$

To express this power in terms of the transverse dielectric function we see that (24) and (34b) give

$$\Delta_{\mathrm{T}}(k^\lambda, x^\lambda) = (1/k_0)^2[2\ \mathrm{Im}\ \varepsilon_{\mathrm{T}}(k^\lambda, x^\lambda)/|\varepsilon_{\mathrm{T}}(k^\lambda, x^\lambda) - \gamma^2|^2)] \tag{37}$$

with $\gamma^2 = (k/k_0)^2$.

Finally

$$dP(x^\lambda)/dk_0\, d\Omega\, d\Sigma = (\hbar c^2/4\pi^4 k_0) \int_0^\infty dk\, k^3[1+2N_T(k^\lambda, x^\lambda)] \times [\mathrm{Im}\, \varepsilon_T(k^\lambda, x^\lambda)/|\varepsilon_T(k^\lambda, x^\lambda) - \gamma^2|^2]. \qquad (38)$$

This expression, valid in the case of local isotropy with slowly varying external currents, exhibits a strong analogy with the equilibrium formula [1].

5. Conclusions

Equations (28) and (38) are the central results of this paper; both of them formally describe the radiation spectrum for non-equilibrium plasma-radiation systems, but their domains of application are quite different.

The first one is a general formula where no assumption of any kind has been made. The presence of tensorial functions such as $N_{\mu\nu}$ and $\Delta_{\mu\nu}$ accounts for the coupling between the transverse and longitudinal fields, which cannot be neglected in the case of a medium subject to a uniform magnetic field or strong density gradients (e.g. astrophysical plasmas or plasma-beam systems). The non-equilibrium state of the charged particles are implicitly contained in the spectral function through the polarisation operator.

On the other hand, (38) describes less extreme situations, where the non-equilibrium inhomogeneities in time and space are slowly varying on a microscopic scale. In a locally isotropic medium, this permits an explicit appearance of the properties of the plasma (through the transverse dielectric function) and leads to a formal analogy with the equilibrium spectrum.

The method exposed in this paper is general (the Green function technique covers the entire region of density and temperature) and may thus be applied in a wide range of plasmas. Of course, it is particularly appropriate for systems where quantum effects are dominating. It is clear that a quantum treatment is necessary to describe rigorously a relativistic plasma; the most evident reason is that for high velocities the distance of closest approach for collisions of two particles becomes smaller than the de Broglie wavelength (the short-ranged collisions must thus be treated quantum mechanically to avoid the divergences for large $\boldsymbol{k}$). Consequently, this work may form a valid basis for the study of radiation in astrophysical plasmas.

In a non-relativistic plasma, when the density becomes sufficiently high, a quasicontinual overlap of wavefunctions may occur (i.e. the particles are 'always' colliding). In such a medium, (28) or (38) provides an accurate description of the radiation since, in addition to their quantum feature, all the collective and screening effects are automatically taken into account in the expansion of $Q_{\mu\nu}$.

As previously noted, all the functions introduced in this work can be evaluated from the quantum kinetic theory of plasmas and radiation related in detail in [5, 6] which include graphical Feynman expansions for $Q_{\mu\nu}$ and Boltzmann-like equations for $N_{\mu\nu}$.

The Green function technique is undoubtedly an accurate method for describing plasma-radiation interaction, but the price to pay for rigour is that it involves rather formidable computational calculations when one exceeds the first-order (collisional) solution of the Dyson equations. In this paper (see the appendix), we limit ourselves to show the consistency of our results with the equilibrium ones.

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Appendix

To describe the radiation spectrum of field fluctuations in an equilibrium system, we simply have to replace the non-equilibrium Green functions by their equilibrium counterparts. This can be achieved through the following changes:

$$\Delta_{\mu\nu}(k^\lambda, R^\lambda) \to \Delta^{\mathrm{eq}}_{\mu\nu}(k^\lambda) \tag{A1a}$$

$$N_{\mu\nu}(k^\lambda, R^\lambda) \to N^{\mathrm{eq}}(k_0) g_{\mu\nu}. \tag{A1b}$$

In the case of an isotropic plasma in thermodynamic equilibrium the above changes become

$$\varepsilon_{\mathrm{L}}(k^\lambda, R^\lambda),\ \varepsilon_{\mathrm{T}}(k^\lambda, R^\lambda) \to \varepsilon^{\mathrm{eq}}_{\mathrm{L}}(k^\lambda),\ \varepsilon^{\mathrm{eq}}_{\mathrm{T}}(k^\lambda) \tag{A2a}$$

$$N_{00}(k^\lambda, R^\lambda),\ N_{\mathrm{T}}(k^\lambda, R^\lambda) \to N^{\mathrm{eq}}(k_0). \tag{A2b}$$

We thus obtain, with the aid of (38),

$$\mathrm{d}P/\mathrm{d}k_0\,\mathrm{d}\Omega\,\mathrm{d}\Sigma = (\hbar c^2/4\pi^4 k_0)\coth(\beta\hbar c k_0/2)\int_0^\infty \mathrm{d}k\,k^3[\mathrm{Im}\,\varepsilon^{\mathrm{eq}}_{\mathrm{T}}(k^\lambda)/|\varepsilon^{\mathrm{eq}}_{\mathrm{T}}(k^\lambda) - \gamma^2|^2] \tag{A3}$$

which is exactly the quantum analogue of the equilibrium formula derived, for example, in [1].

As previously noted, the convergence (for large $\boldsymbol{k}$) of this integral remains subject to a careful quantum mechanical treatment of close collisions [11, 12]. Of course, in the case of weak damping, the above problem disappears and the integral in (A3) is always convergent.

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